Madras College Maths Department
Higher Maths
Apps 1.3 Recurrence Relations

Written solutions for each exercise are available at

http://madrasmaths.com/courses/higher/revison_materials_higher.html

You should check your solutions at the end of each exercise and ask your teacher or attend study support if there any problems.

Introduction to Sequences

A **sequence** is an ordered list of objects (usually numbers).

Usually we are interested in sequences which follow a particular pattern. For example, $1,2,3,4,5,6,\ldots$ is a sequence of numbers – the "…" just indicates that the list keeps going forever.

Writing a sequence in this way assumes that you can tell what pattern the numbers are following but this is not always clear, e.g.

$$28,\ 22,\ 19,\ 17\tfrac{1}{2},\ \ldots .$$

For this reason, we prefer to have a formula or rule which explicitly defines the terms of the sequence.

It is common to use subscript numbers to label the terms, e.g.

$$u_1,\ u_2,\ u_3,\ u_4,\ \ldots$$

so that we can use u_n to represent the nth term.

We can then define sequences with a formula for the nth term. For example:

Formula	List of terms
$u_n = n$	$1,\ 2,\ 3,\ 4,\ \ldots$
$u_n = 2n$	$2,\ 4,\ 6,\ 8,\ \ldots$
$u_n = \frac{1}{2}n(n+1)$	$1,\ 3,\ 6,\ 10,\ \ldots$
$u_n = \cos\left(\frac{n\pi}{2}\right)$	$0,\ -1,\ 0,\ 1,\ \ldots$

Notice that if we have a formula for u_n, it is possible to work out *any* term in the sequence. For example, you could easily find u_{1000} for any of the sequences above without having to list all the previous terms.

Question 1 - Given that $u_{n+1} = 3u_n + 7$ and $u_0 = 3$ calculate u_1, u_2 and u_3.

Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

Linear Recurrence Relations

In Higher, we will deal with recurrence relations of the form

$$u_{n+1} = au_n + b$$

where a and b are any real numbers and u_0 is specified. These are called **linear recurrence relations** of order one.

Note

To properly define a sequence using a recurrence relation, we must specify the initial value u_0.

EXAMPLES

1. A patient is injected with 156 ml of a drug. Every 8 hours, 22% of the drug passes out of his bloodstream. To compensate, a further 25 ml dose is given every 8 hours.

   (a) Find a recurrence relation for the amount of drug in his bloodstream.

   (b) Calculate the amount of drug remaining after 24 hours.

Q) A warehouse restocks by ordering 5% of the existing stock at the beginning of each month. During each month 1000 items are distributed from the warehouse. If the existing stock is 5000 items:

a) Find a recurrence relation for the amount of items in stock.

b) Calculate the expected number of items in stock after 4 months.

2. A sequence is defined by the recurrence relation $u_{n+1} = 0{\cdot}6u_n + 4$ with $u_0 = 7$.
   Calculate the value of u_3 and the smallest value of n for which $u_n > 9{\cdot}7$.

Using a Calculator

Using the ANS button on the calculator, we can carry out the above calculation more efficiently.

						7	=
0	.	6	×	ANS	+	4	=
							=
							=

Finding a Recurrence Relation for a Sequence

If we know that a sequence is defined by a linear recurrence relation of the form $u_{n+1} = au_n + b$, and we know three consecutive terms of the sequence, then we can find the values of a and b.

This can be done easily by forming two equations and solving them simultaneously.

EXAMPLE

A sequence is defined by $u_{n+1} = au_n + b$ with $u_1 = 4$, $u_2 = 3{\cdot}6$ and $u_3 = 2{\cdot}04$.

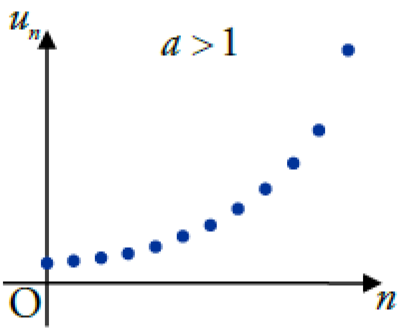
Find the values of a and b.

Divergence and Convergence

If we plot the graphs of some of the sequences that we have been dealing with, then some similarities will occur.

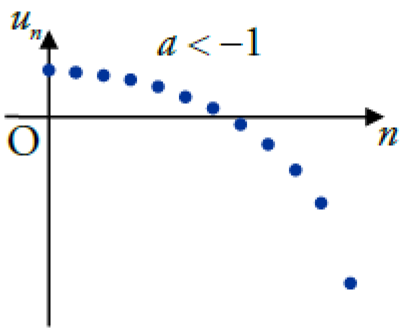
Divergence

Sequences defined by recurrence relations in the form $u_{n+1} = au_n + b$ where $a < -1$ or $a > 1$, will have a graph like this:



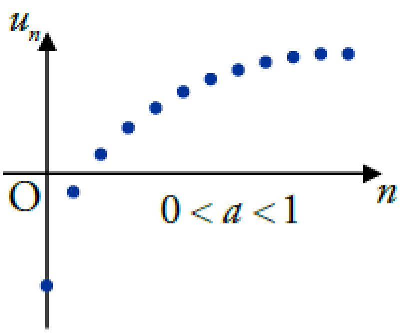
Sequences like this will continue to increase or decrease forever.

They are said to **diverge**.



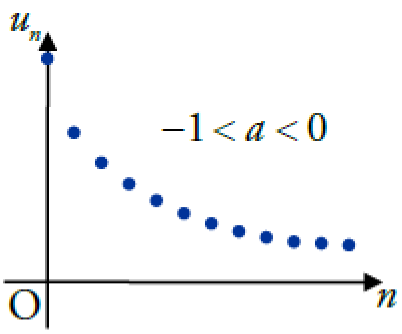
Convergence

Sequences defined by recurrence relations in the form $u_{n+1} = au_n + b$ where $-1 < a < 1$, will have a graph like this:



Sequences like this "tend to a limit".

They are said to **converge**.



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The Limit of a Sequence

We saw that sequences defined by $u_{n+1} = au_n + b$ with $-1 < a < 1$ "tend to a limit". In fact, it is possible to work out this limit just from knowing a and b.

The sequence defined by $u_{n+1} = au_n + b$ with $-1 < a < 1$ tends to a limit l as $n \to \infty$ (i.e. as n gets larger and larger) given by

$$l = \frac{b}{1-a}.$$

You will need to know this formula, as it is not given in the exam.

EXAMPLES

1. The deer population in a forest is estimated to drop by 7·3% each year. Each year, 20 deer are introduced to the forest. The initial deer population is 200.

   (a) How many deer will there be in the forest after 3 years?

   (b) What is the long term effect on the population?

2. A sequence is defined by the recurrence relation $u_{n+1} = ku_n + 2k$ and the first term is u_0.
   Given that the limit of the sequence is 27, find the value of k.

Practice Unit Assessments

Practice 1

1 A sequence is defined by the recurrence relation $u_{n+1} = mu_n + c$ where m and c are constants.
It is known that $u_1 = 3$, $u_2 = 7$, and $u_3 = 23$
Find the recurrence relation described by the sequence and use it to find the value of u_5.

(4)

2 On a particular day at 09:00, a doctor injects a first dose of 400 mg of medicine into a patient's bloodstream. The doctor then continues to administer the medicine in this way at 09:00 each day.

The doctor knows that at the end of the 24-hour period after an injection, the amount of medicine in the bloodstream will only be 11% of what it was at the start.

(a) Set up a recurrence relation which shows the amount of medicine in the bloodstream immediately after an injection.

(1)

The patient will overdose if the amount of medicine in their bloodstream exceeds 500 mg.

(b) In the long term, if a patient continues with this treatment, is there a danger they will overdose?
Explain your answer.

(2 + #2.2)

Practice 2

1 On a particular day at 07:00, a vet injects a first dose of 65 mg of medicine into a dog's bloodstream. The vet then continues to administer the medicine in this way at 07:00 each day.

The vet knows that at the end of the 24-hour period after an injection, the amount of medicine in the bloodstream will only be 18% of what it was at the start.

(a) Set up a recurrence relation which shows the amount of medicine in the bloodstream immediately after an injection.

The dog will overdose if the amount of medicine in its bloodstream exceeds 85 mg.

(b) In the long term, if the dog continues with this treatment, is there a danger it will overdose?
Explain your answer.

(3 +#2.2)

2 A sequence is generated by the recurrence relation $u_{n+1} = 2k\,u_n + 5$, $u_{n+1} = 3ku_n + 7$ and $u_0 = 3$.

For what values of k does this sequence have a limit as $n \to \infty$? (1)

Practice Unit Assessment Solutions

Recurrence Relations UAs

Test 1

(1) $u_2 = mu_1 + c \quad u_3 = mu_2 + c$

$7 = 3m + c$ (1) $\quad 23 = 7m + c$ (2)

$16 = 4m$ ((2) − (1))

$m = 4$

sub in (1) $7 = 3(4) + c$

$c = -5$

(2)(a) $u_{n+1} = 0{\cdot}11u_n + 400$ need u_0 $u_0 = 400$

(b) $L = \frac{b}{1-a}$ A limit exists as $a = 0{\cdot}11$ and $-1 < 0{\cdot}11 < 1$

$= \frac{400}{0{\cdot}89}$

$= 449.44$ (2dp) ml

No as $449.44 < 500$.

Test 2

(1)(a) $u_{n+1} = 0{\cdot}18u_n + 65$

$u_0 = 65$

(b) $L = \frac{b}{1-a}$ A limit exists as $a = 0{\cdot}18$, $-1 < 0{\cdot}18 < 1$

$= \frac{65}{1-0{\cdot}18}$

$= 79.27$ mg (2dp)

No as $79.27 < 85$

(2) $u_{n+1} = 3k\,u_n + 7$

$u_{n+1} = a\,u_n + b$

A limit exists if $-1 < a < 1$

so $-1 < 3k < 1$

$-\frac{1}{3} < k < \frac{1}{3}$

Test 1 Q1 continued

$U_{n+1} = 4u_n - 5$

$U_4 = 4(23) - 5$
$= 87$

$U_5 = 4(87) - 5$
$= 343$